Math 10B with Professor Stankova
Worksheet, Discussion #12; Tuesday, 2/27/2018
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1 Probability Distributions

1.1 Concepts

1.

Distribution	PMF	Example
Uniform	If $\#R(X) = n$, then $f(x) = \frac{1}{n}$ for all $x \in R(X)$.	Dice roll, $f(1) = f(2) = \cdots = f(6) = \frac{1}{6}$.
Bernoulli Trial	$f(0) = 1 - p$, $f(1) = p$	Flipping a biased coin
Binomial	$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$	p is probability of success. Repeat n Bernoulli trials. Number of 6's rolled when rolling 10 die is $f(k) = \binom{10}{k}(1/6)^k(5/6)^{10-k}$.
Geometric	$f(k) = (1-p)^k p$	k failures and then a success.
Hyper-Geometric	$f(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$	Counting the number of red balls I pick out of n balls drawn if there are m red balls out of N balls total.
Poisson	$f(k) = \frac{\lambda^k e^{-\lambda}}{k!}$	Count the number of babies born today if on average there are 3 babies born a day.

1.2 Examples

2. I am picking cards out of a deck. What is the probability that I pull out 1 heart out of 5 cards if I pull with replacement? If I pull 5 cards at once?

Solution: With replacement is repeated Bernoulli trials which means binomial distribution. The probability of a success or pulling out a heart is $\frac{1}{4}$. Therefore, the probability of pulling 1 heart out of 5 is

$$\binom{5}{1}\frac{1}{4^1} \cdot \frac{3^{5-1}}{4^{5-1}}.$$

If do not have replacement, then this is a hyper-geometric distribution with $N = 52, n = 5, m = 13$, the the answer is

$$\frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}}.$$

3. What is the probability that first heart is the third card I draw (with replacement)?

> **Solution:** We want to know the probability of the first success, which is geometric since we are doing with replacement. The probability of a success is $p = \frac{1}{4}$ and we have two failures before we have a success so $k = 2$. Hence the answer is $f(2) = (3/4)^2(1/4) = \frac{9}{64}$.

1.3 Problems

4. True **FALSE** We cannot talk about Bernoulli trials for rolling a 5 because there are 6 outputs and we need 2 for a Bernoulli trial.

> **Solution:** We can change this into a Bernoulli trial by saying the probability of success is 1/6 and all other outputs are failures, which makes it into success and failure.

5. True **FALSE** The geometric distribution, like the hyper-geometric distribution, assumes that the trials are dependent (without replacement).

> **Solution:** The geometric distribution has independent Bernoulli trials but the hypergeometric distribution has dependent ones because we pick without replacement.

6. In a class of 50 males and 80 females, I give out 3 awards randomly. What is the probability that females will win all 3 awards if the awards must go to different people? What about if the same person can win all three awards?

> **Solution:** This is like the probability of picking 3 females out of 3 people chosen. If the awards must go to different people, there is no replacement so it is the hypergeometric distribution where a success is picking a female. So we have $N = 130$ students total, there are $m = 80$ females, and I am picking $n = 3$ students and I want $k = 3$ females. This gives
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> $$f(3) = \frac{\binom{80}{3}\binom{50}{0}}{\binom{130}{3}} = \frac{\binom{80}{3}}{\binom{130}{3}}.$$
>
> If the same person can will all the awards, then we are choosing with replacement. So, this is a binomial distribution where the probability of success is $p = \frac{80}{130}$. Thus, we have that the answer is
>
> $$f(3) = \binom{3}{3}\left(\frac{80}{130}\right)^3\left(\frac{50}{130}\right)^0 = \frac{80^3}{130^3}.$$

7. At Berkeley, there is an equal number of people aged $18, 19, \ldots, 27$. I cold call someone at random and ask for their age. What is the PMF for their age? Suppose that undergraduates are aged 18 though 21 inclusive. What is the probability that I have to call 10 people until I call an undergraduate (the undergraduate is the 10th person I call)? What is the probability that I call 4 undergraduates out of 10 people I call (if I can call someone more than once)?

Solution: This is a uniform distribution on $[18, 27]$. There are 10 numbers in between and hence

$$f(k) = \begin{cases} \frac{1}{10} & 18 \leq x \leq 27 \\ 0 & \text{otherwise} \end{cases}.$$

The probability of calling an undergrad is $f(18) + f(19) + f(20) + f(21) = \frac{4}{10} = \frac{2}{5}$. The probability that the first undergrad I call is the 10th person is given by the geometric distribution since we are talking about times until a success. I have 9 failures before and so this is

$$f(9) = (1 - 2/5)^9 (2/5) = \frac{3^9 \cdot 2}{5^{10}}.$$

The probability that I call 4 undergraduates out of 10 people is given by a binomial distribution since I can call someone more than once and hence there is replacement. So plugging this into the binomial distribution gives

$$f(4) = \binom{10}{4} (2/5)^4 (3/5)^6.$$

8. For a lottery, 6 distinct numbers are drawn out of 60 and to win, you need to match all 6 numbers. What is the probability that I win? If I buy 100 different tickets, what is the probability that I win?

Solution: This can be thought of as there are a total of $\binom{60}{6}$ different lottery tickets and only one of them is a winner, or only one of them is tagged. When I buy different tickets, I am picking without replacement so this is the hyper-geometric distribution. There are $N = \binom{60}{6}$ total tickets and $m = 1$ of them is a success. Then out of the n balls I draw, I want $k = 1$ success. For the first case, if I only pick one ticket, then $n = 1$ and we get

$$f(1) = \frac{\binom{1}{1}\binom{\binom{60}{6}-1}{0}}{\binom{\binom{60}{6}}{1}} = \frac{1}{\binom{60}{6}}.$$

If I pick 100 tickets, then $n = 100$ and we get

$$f(1) = \frac{\binom{1}{1}\binom{\binom{60}{6}-1}{99}}{\binom{\binom{60}{6}}{100}}.$$

1.4 Extra Problems

9. In a class of 80 males and 60 females, I give out 3 awards randomly. What is the probability that 2 females will win awards if the awards must go to different people? What about if the same person can win all three awards?

Solution: This is like the probability of picking 2 females out of 3 people chosen. If the awards must go to different people, there is no replacement so it is the hypergeometric distribution where a success is picking a female. So we have $N = 140$ students total, there are $m = 60$ females, and I am picking $n = 3$ students and I want $k = 2$ females. This gives

$$f(2) = \frac{\binom{60}{2}\binom{80}{1}}{\binom{140}{3}}.$$

If the same person can will all the awards, then we are choosing with replacement. So, this is a binomial distribution where the probability of success is $p = \frac{60}{140}$. Thus, we have that the answer is

$$f(2) = \binom{3}{2}\left(\frac{60}{140}\right)^2\left(\frac{80}{140}\right)^1.$$

10. At Berkeley, there is an equal number of people aged $18, 19, \ldots, 27$. I cold call someone at random and ask for their age. What is the PMF for their age? Suppose that undergraduates are aged 18 though 21 inclusive. What is the probability that I have to call 30 people until I call an undergraduate (the undergraduate is the 30th person I call)? What is the probability that I call 8 undergraduates out of 15 people I call (if I can call someone more than once)?

Solution: This is a uniform distribution on $[18, 27]$. There are 10 numbers in between and hence

$$f(k) = \begin{cases} \frac{1}{10} & 18 \leq x \leq 27 \\ 0 & \text{otherwise} \end{cases}.$$

The probability of calling an undergrad is $f(18) + f(19) + f(20) + f(21) = \frac{4}{10} = \frac{2}{5}$. The probability that the first undergrad I call is the 30th person is given by the geometric distribution since we are talking about times until a success. I have 29 failures before so this is

$$f(29) = (1 - 2/5)^{29}(2/5).$$

The probability that I call 8 undergraduates out of 15 people is given by a binomial distribution since I can call someone more than once and hence there is replacement.

So plugging this into the binomial distribution gives

$$f(8) = \binom{15}{8}(2/5)^8(3/5)^7.$$

11. For a lottery, 4 distinct numbers are drawn out of 40 and to win, you need to match all 4 numbers. What is the probability that I win? If I buy 30 different tickets, what is the probability that I win?

Solution: This can be thought of as there are a total of $\binom{40}{4}$ different lottery tickets and only one of them is a winner, or only one of them is tagged. When I buy different tickets, I am picking without replacement so this is the hyper-geometric distribution. There are $N = \binom{40}{4}$ total tickets and $m = 1$ of them is a success. Then out of the n balls I draw, I want $k = 1$ success. For the first case, if I only pick one ticket, then $n = 1$ and we get

$$f(1) = \frac{\binom{1}{1}\binom{\binom{40}{4}-1}{0}}{\binom{\binom{40}{4}}{1}} = \frac{1}{\binom{40}{4}}.$$

If I pick 30 tickets, then $n = 30$ and we get

$$f(1) = \frac{\binom{1}{1}\binom{\binom{40}{4}-1}{29}}{\binom{\binom{40}{4}}{30}}.$$